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A Three Parameter Lifetime Distribution with Increasing, Decreasing and Bathtub Shaped Failure Rates

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ABSTRACT
In this paper, we propose an extension to the existing reduced Kies distribution called the extended reduced Kies (ExRKD) distribution. We study the properties of the ExRKD and utilize it on real-world COVID-19 datasets to estimate parameters with the help of maximum likelihood estimation (MLE) procedures. Also perform simulation studies to evaluate the asymptotic behaviour of the MLEs, offering insights into their performance and reliability.

KEYWORDS
Beta Weibull distribution; COVID-19; Kies distribution; maximum likelihood estimation; model selection; simulation.

1. Introduction

Kumar and Dharmaja (2013) studied a particular form of the Kies distribution through the name “the reduced Kies distribution(RKD)” and it has the following probability density function (p.d.f.).

$$f\left(x\right)=\frac{\beta x^{\beta-1}\exp\left\{-\left(\frac{x}{1-x}\right)^{\beta}\right\}}{\left(1-x\right)^{\beta+1}} \tag{1}$$

where $\beta > 0$ and $x \in (0, 1)$. The RKD can be viewed as a functional form of the well-known Weibull distribution and it has been found extensive applications in several areas, particularly in certain situations where the Weibull model fails to give better fits. It enjoys several properties similar to that of Weibull distribution. The cumulative distribution function (c.d.f.) of the RKD has an interesting property which observed in the c.d.f. plot where all curves intersect at (0.5, exp (-1) point. The Hazard rate function is either decreasing or increasing based on its values of the parameters. It is a decreasing function for $x < \frac{1-\beta}{2}$ provided $\beta > 1$ and increasing function of x if $\beta > 1$ or $x > \frac{1-\beta}{2}$ with $\beta < 1$.The RKD is useful to studying data sets involving survival

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rates, incidence rates, and death rates. Kumar and Dharmaja (2017) introduced an exponentiated version of the RKD and named it as "the exponentiated reduced Kies distribution". They investigated several characteristics of it and shown that it will be suitable for modelling data sets with increasing or decreasing hazard rates.

Through the present paper we proposed a general class of RKD for modelling complex data sets which possess increasing, decreasing and bathtub shaped hazard functions. The proposed class we call "the extended reduced Kies distribution (or in short, the ExRKD)". In section 2, we provide the definition of the ExRKD along with derivation of some of its properties such as expressions for raw moments, mean, variance, coefficient of skewness and coefficient of kurtosis. Section 3 focuses on estimating the ExRKD parameters, while in section 4 and 5 we examine certain application of the ExRKD utilizing COVID data sets. Further we evaluate the efficiency of maximum likelihood estimators (MLEs) through a simulation study.

2. Definition and Properties

In this section, we define the ExRKD and derive some of its statistical properties.
Definition 2.1. *A continuous random variable X is said to have the extended reduced Kies distribution ($ExRKD$) if its p.d.f. is expressed in the following form, in which $\beta > 0$, $\rho > 0$, $x \in (0,1)$ and $\omega \in [0,1]$.*

$$g(x) = g(x; \beta, \rho, \omega)$$

$$= \frac{1}{x(1-x)}\left[\omega\beta\left(\frac{x}{1-x}\right)^{\beta} e^{-\left(\frac{x}{1-x}\right)^{\beta}} + (1-\omega)\rho\beta\left(\frac{x}{1-x}\right)^{\rho\beta} e^{-\left(\frac{x}{1-x}\right)^{\rho\beta}}\right] \tag{2}$$

$$= \frac{1}{x(1-x)}\left[\omega\beta u_1 e^{-u_1} + (1-\omega)\rho\beta u_2 e^{-u_2}\right],$$

where $u_1 = \left(\frac{x}{1-x}\right)^{\beta}$ and $u_2 = \left(\frac{x}{1-x}\right)^{\rho\beta}$. Clearly, when $\rho = 1$, the p.d.f. (2) reduces to the p.d.f. of the RKD as given in (1). Now we obtain the c.d.f., survival Function, hazard function and an expression for the raw moments of the ExRKD through the following results.
Result 2.1. *The c.d.f. F(x) of the ExRKD is the following, for $x \in (0,1)$.*

$$F(x) = \omega\left(1 - e^{-u_1}\right) + (1-\omega)\left(1 - e^{-u_2}\right)$$

Result 2.2. *The survival Function $S(x)$ and hazard function $h(x)$ of the ExRKD are respectively given by*

$$S(x) = \omega e^{-u_1} + (1-\omega)e^{-u_2}$$

and

$$h(x) = \frac{\frac{1}{x(1-x)}\left[\omega\beta u_1 e^{-u_1} + (1-\omega)\rho\beta u_2 e^{-u_2}\right]}{\omega e^{-u_1} + (1-\omega)e^{-u_2}}.$$

The proofs of Results 2.1 and 2.2 are straightforward and hence omitted. Figures 1,2 and 3 displays the p.d.f., c.d.f. and hazard function plots of the ExRKD for particular parameter values.

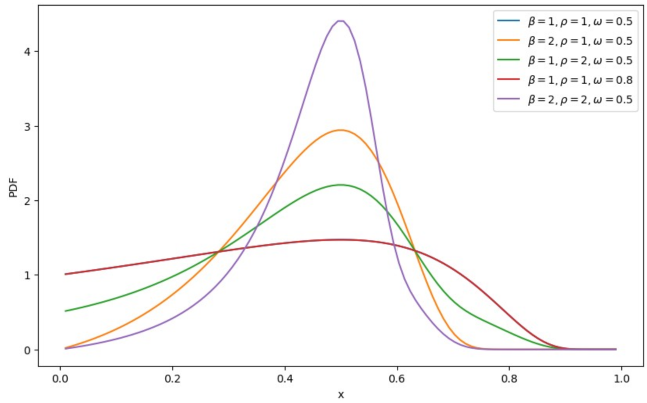

Figure 1.: The p.d.f. plot of the ExRKD for different parameters.

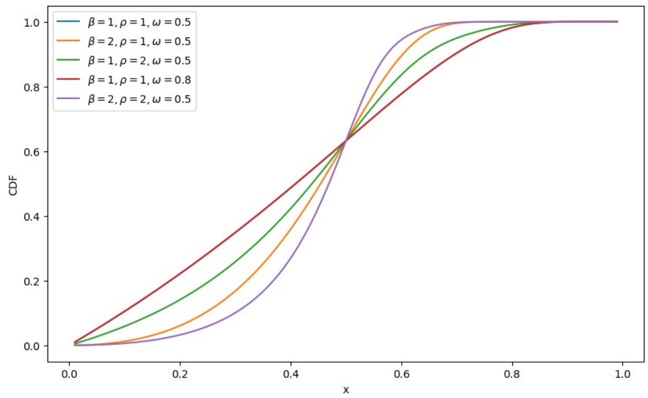

Figure 2.: The c.d.f. plot of the ExRKD for different parameters.

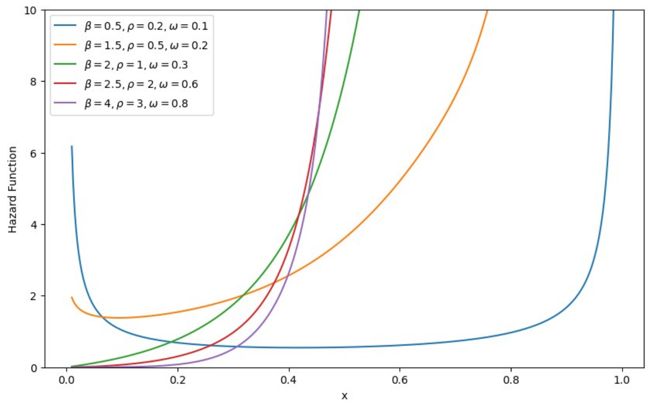

Figure 3.: Plots of the hazard function of the ExRKD for different parameters.

Result 2.3. *If X follows the ExRKD with p.d.f. (2), then the r^{th} raw moment μ_r' is the following.*

$$\mu_r' = \sum_{j=0}^{r} \binom{r}{j} (-1)^j \left[\omega\ \delta_j(\beta,1) + (1-\omega)\,\delta_j(\beta,\rho)\right], \tag{3}$$

where $\delta_j(\beta,\rho) = \int_0^\infty e^{-u_j}\left(1 + u_j{}^{1/\rho\beta}\right)^{-j} du_j$ *for* $j = 0, 1, 2, \ldots, r.$

Proof: By definition, the r^{th} raw moment μ_r' of the ExRKD is

$$\mu_r' = \int_0^1 x^r \frac{1}{x(1-x)} \left[\omega\beta\left(\frac{x}{1-x}\right)^{\beta} e^{-\left(\frac{x}{1-x}\right)^{\beta}} + (1-\omega)\,\rho\beta\left(\frac{x}{1-x}\right)^{\rho\beta} e^{-\left(\frac{x}{1-x}\right)^{\rho\beta}}\right] dx \tag{4}$$

$$= I_1 + I_2,$$

where

$$I_1 = \int_0^1 x^r \frac{1}{x(1-x)} \left[\omega\beta\left(\frac{x}{1-x}\right)^{\beta} e^{-\left(\frac{x}{1-x}\right)^{\beta}}\right] dx \tag{5}$$

and

$$I_2 = \int_0^1 x^r \frac{1}{x(1-x)} \left[(1-\omega)\,\rho\beta\left(\frac{x}{1-x}\right)^{\rho\beta} e^{-\left(\frac{x}{1-x}\right)^{\rho\beta}}\right] dx. \tag{6}$$

If we put $u_1 = \left(\frac{x}{1-x}\right)^{\beta}$ in (4), we get

$$I_1 = \omega \int_0^\infty \frac{u_1^{r/\beta}}{\left(u_1^{1/\beta}+1\right)^r}\ e^{-u_1} du_1$$

$$= \omega \int_0^\infty \left(1 - \frac{1}{1+u_1^{1/\beta}}\right)^r e^{-u_1} du_1$$

Now, by binomial expansion we obtain the following.

$$I_1 = \omega \sum_{j=0}^{r} \binom{r}{j} (-1)^j \int_0^\infty e^{-u_1}\left(u_1^{1/\beta}+1\right)^{-j} du_1 \tag{7}$$

Similarly, if we put $u_2 = \left(\frac{x}{1-x}\right)^{\rho\beta}$ in (4) to get

$$I_2 = (1-\omega) \sum_{j=0}^{r} \binom{r}{j} (-1)^j \int_0^\infty e^{-u_2}\left(u_2^{1/\rho\beta}+1\right)^{-j} du_2. \tag{8}$$

Now on substituting (7) and (8) in (4), we get (3). By using (3), we calculate the mean, variance, coefficients of skewness and coefficients of kurtosis of the ExRKD for particular values of its parameters and listed them in Tables 1 and 2.

Table 1.: For particular values of the parameters, the computed values of mean, variance, coefficients of skewness, and coefficients of kurtosis of the ExRKD.

β	$\rho = 0.1,\ \omega = 0.5$			
	Mean	**Variance**	γ_1	γ_2
0.1	0.366	0.213	0.546	-1.637
0.5	0.373	0.155	0.511	-1.396
0.75	0.38	0.136	0.484	-1.255
1	0.386	0.123	0.455	-1.128
1.5	0.396	0.105	0.39	-0.925
2	0.403	0.092	0.32	-0.771
2.5	0.409	0.083	0.246	-0.646
3	0.413	0.075	0.172	-0.54

Table 2.: For particular values of the parameters, the computed values of mean, variance, coefficients of skewness, and coefficients of kurtosis of the ExRKD.

β	$\rho = 0.5,\ \omega = 0.2$			
	Mean	**Variance**	γ_1	γ_2
0.1	0.368	0.211	0.546	-1.631
0.5	0.371	0.139	0.48	-1.392
0.75	0.377	0.108	0.397	-1.287
1	0.384	0.087	0.297	-1.204
1.5	0.398	0.059	0.089	-1.043
2	0.41	0.042	0.1	-0.851
2.5	0.42	0.032	0.261	-0.63
3	0.43	0.025	0.396	-0.386

3. Estimation

Let $X_1,\ X_2,\ X_3,\ \ldots,\ X_n$ be a random sample from an ExRKD with the parameters β, ρ and ω. Then the log likelihood function for the ExRKD is given by

$$l = -\sum_{i=1}^{n}(x_i) - \sum_{i=1}^{n}(1-x_i) + \sum_{i=1}^{n}\ln\left(\omega\beta\left(\frac{x_i}{1-x_i}\right)^{\beta} e^{-\left(\frac{x_i}{1-x_i}\right)^{\beta}} + (1-\omega)\,\rho\beta\left(\frac{x_i}{1-x_i}\right)^{\rho\beta} e^{-\left(\frac{x_i}{1-x_i}\right)^{\rho\beta}}\right).$$

Now, by differentiating the log likelihood function with respect to the parameters of the ExRKD and equating to zero, we derive the following likelihood equations, in which

$$\lambda_i\left(m,k,\beta\right) = \left(\frac{x_i}{1-x_i}\right)^{m\beta} e^{-\left(\frac{x_i}{1-x_i}\right)^{\beta}} \left[ln\left(\frac{x_i}{1-x_i}\right)\right]^{k}$$

for m=1, 2 and k=0, 1.

$$\frac{\partial l}{\partial \beta} = 0$$

equivalently

$$\sum_{i=1}^{n}\left[\frac{\omega\lambda_i\left(1,0,\beta\right) + \omega\beta\left[\lambda_i\left(1,1,\beta\right) - \lambda_i\left(2,1,\beta\right)\right] + \rho\left(1-\omega\right)\lambda_i\left(1,0,\rho\beta\right) + \rho\beta\left(1-\omega\right)\left[\lambda_i\left(1,1,\rho\beta\right) - \lambda_i\left(2,1,\rho\beta\right)\right]}{\omega\beta\lambda_i\left(1,0,\beta\right) + \left(1-\omega\right)\rho\beta\lambda_i\left(1,0,\rho\beta\right)}\right] = 0 \tag{9}$$

$$\frac{\partial l}{\partial \rho} = 0$$

equivalently

$$\sum_{i=1}^{n}\left[\frac{\left(1-\omega\right)\lambda_i\left(1,0,\rho\beta\right) + \rho\left(1-\omega\right)\left[\lambda_i\left(1,1,\rho\beta\right) - \lambda_i\left(2,1,\rho\beta\right)\right]}{\omega\beta\lambda_i\left(1,0,\beta\right) + \left(1-\omega\right)\rho\beta\lambda_i\left(1,0,\rho\beta\right)}\right] = 0 \tag{10}$$

and

$$\frac{\partial l}{\partial \omega} = 0$$

equivalently,

$$\sum_{i=1}^{n}\left[\frac{\beta\lambda_i\left(1,0,\beta\right) - \rho\beta\lambda_i\left(1,0,\rho\beta\right)}{\omega\beta\lambda_i\left(1,0,\beta\right) + \left(1-\omega\right)\rho\beta\lambda_i\left(1,0,\rho\beta\right)}\right] = 0\ . \tag{11}$$

On solving likelihood equations (9), (10) and (11) we get the MLEs of the parameters β, ρ and ω of the ExRKD. It is not possible to obtain the exact distribution MLEs of the unknown parameters β, ρ, and ω of the ExRKD , since they are not in closed form expressions. Hence, we derived the second order partial derivatives of the log likelihood function with respect to respective parameters and verified that their values are negative which indicates the existence of the maximum likelihood estimates for the respective parameters of the ExRKD.

4. Applications

In this section we discuss some applications of the ExRKD utilizing following COVID-19 data sets:

Data Set 1: This dataset contains COVID-19 mortality data for Canada over a period of 56 days, from November 1 to December 26, 2020. The data is sourced from the World Health Organization (WHO). [`https://covid19.who.int/`]

Data Set 2: This dataset includes the total number of COVID-19 deaths per million people in India from April 10, 2020, to May 3, 2020. [`https://covid.ourworldindata.org/data/owid-covid-data.csv`]

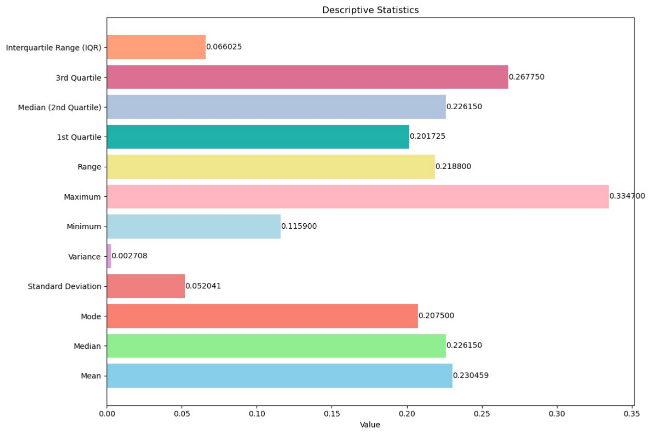

Figure 4.: Bar chart of descriptive statistics for data set 1.

Figure 4, shows the descriptive statistics for the data set 1, which gives insights into the distribution and variability of the data set 1. The mean is 0.2304 and median is 0.2261, the difference between the two is relatively insignificant, which means the distribution is quite symmetric. The range is 0.218, the difference between the minimum (0.1159) and maximum (0.3347) values. The interquartile range(IQR) is 0.0660, representing the range of the middle 50% of the data. The standard deviation is 0.0520, indicating moderate variability around the mean. The mode is 0.2075, the most common value in the data set 1.

In figure 5, the histogram shows that the data is symmetrical and most of the values are located between 0.15 and 0.30. Most frequently, the values are distributed in the range from 0.20 to 0.25. The box plot shows the median, quartiles, and possible outliers for the dataset. The median of the data is around 0.226, whereas its interquartile range spans approximately from 0.20 to 0.27. No clear outliers because the whiskers stretch to include both the minimum and maximum without any points beyond. In QQ plot points are close to the red diagonal line and hence the data normally distributed. The last one is violin plot, which combines a box-plot and a kernel density plot. This provides an idea of how the data is spread over different values. The width of the violin plot for

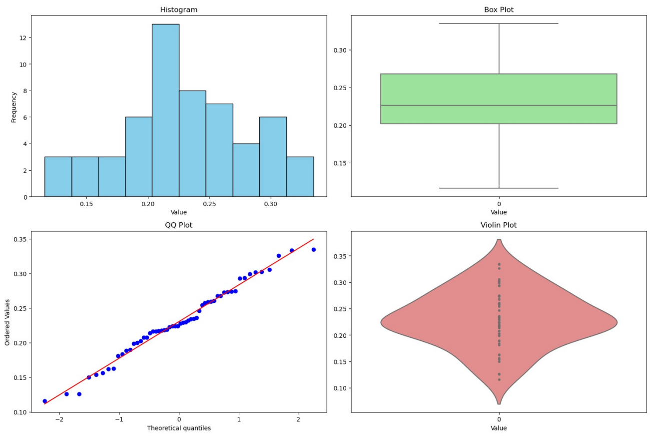

Figure 5.: Histogram, box plot, QQ plot and Violin plot for data set 1.

different values shows data density. Thus, the distribution looks to be symmetric and bell-shaped.

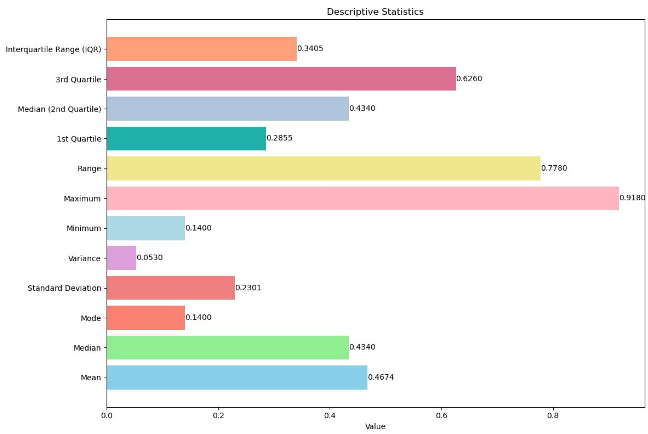

Figure 6.: Bar chart of descriptive statistics for data set 2.

The Figure 6, shows bar graph of the important descriptive statistics for the data set 2. The distribution seems to be slightly right-skewed since the mean and median are respectively 0.4674 and 0.4340. The range for the dataset varies from 0.1400 to 0.9180, which is equal to 0.7780, and ranges in a similar way that the standard deviation of 0.2301 differs. The IQR equals 0.3405, which represents the variation of the middle 50% of the data. The mode is 0.1400 which is the most common value. From Figure 7, the box plot suggests that the data is symmetrically distributed around the median with no significant outliers. The Q-Q plot indicates that the data follows a normal distribution with slight deviations at the tails. The violin plot provides a view of the data's density and distribution.

All the three test results in Table 3 shows that the ExRKD distribution is good fit for both data sets because of high p-values for the Kolmogorov-Smirnov test. The test statistic values of Cramér-Von Mises and Anderson-Darling tests are relatively low, which indicates that there is no significant evidence to reject the null hypothesis. Thus, the ExRKD provides good fit to both data sets considered here. Now for model comparison, we considered reduced Kies distribution (RKD), exponentiated reduced Kies distribution (ERKD) and beta Weibull (BW) in Tables 4 and 5.

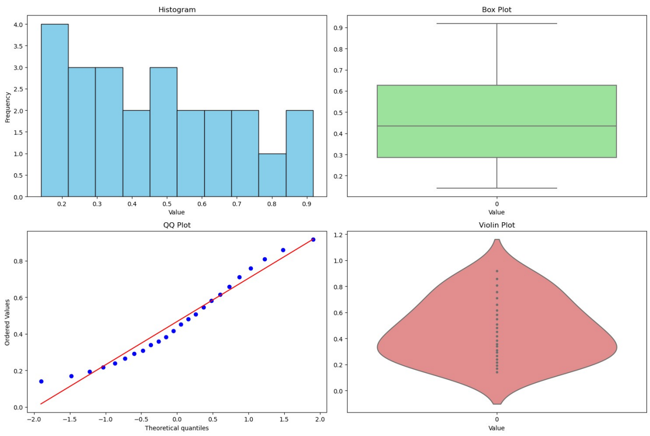

Figure 7.: Histogram, box plot, QQ plot and Violin plot for data set 2.

Table 3.: K-S statistics, Cramér-Von Mises and Anderson–Darling test for ExRKD corresponding to Data Sets 1 and 2

Dataset	Kolmogorov-Smirnov test	P-value	Cramer-Von Mises Test	Anderson-Darling Test
1	0.125	0.775	0.072	0.666
2	0.166	0.902	0.144	1.031

Table 4.: Fitting of various models to data set 1

Model	Estimates	Log-likelihood	AIC	BIC
RKD	$\beta = 0.282$	-94.486	190.972	192.998
ERKD	$\delta = 5.101$, $\beta = 0.452$	-57.693	119.387	123.438
BW	$c = 0.257$, $\alpha = 0.481$, $\beta = 0.644$, $\gamma = 0.793$	-79.210	166.421	174.523
ExRKD	$\beta = 1.089$, $\rho = 1.00$, $\omega = 0.583$	12.783	-19.5662	-13.4901

From Tables 4 and 5, one can observe the values of log-likelihood, Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC) for different models. These metrics help to assess model fit and complexity and guiding in selecting the best model for the data sets. Hence the results reveal that the ExRKD fitted as best model for both data sets because of the lowest AIC and BIC values.

Table 5.: Fitting of various models to data set 2

Model	**Estimates**	**Log-likelihood**	**AIC**	**BIC**
RKD	$\beta = 0.432$	-43.937	89.875	91.053
ERKD	$\delta = 2.159$, $\beta = 0.417$	-38.367	80.734	83.090
BW	$c = 0.032$, $\alpha = 0.426$, $\beta = 0.609$, $\gamma = 0.803$	-37.561	83.122	87.834
ExRKD	$\beta = 1.426$, $\rho = 0.426$, $\omega = 0.551$	0.297	5.405	8.939

5. Simulation

This section presents a simulation study aimed at evaluating the asymptotic behaviour of the MLEs for the parameters of the ExRKD.

Table 6.: Result of simulation study for the ExRKD

Sample size n	**Parameter**	**Average Bias**	**MSE**
10	ρ	-0.0518	0.0268
	β	-0.1377	0.18964
	ω	-0.02280	0.005202
25	ρ	-0.0201	0.01012
	β	-0.06324	0.1000
	ω	0.004799	0.000575
50	ρ	-0.0103	0.0053
	β	-0.02665	0.03551
	ω	0.002359	0.0002783
100	ρ	-0.00469	0.0022
	β	-0.01176	0.01384
	ω	-0.000182	3.32607×10^{-6}

The simulation study aimed at determining the properties of the likelihood estimators for the parameters of the ExRKD were based on n observations from the considered sets of parameters $\beta = 2$, $\rho = 0.9$ and $\omega = 0.5$. A total of 200 samples were considered for this study to evaluate the performance of the various MLEs of the parameters of the ExRKD, with reference to their mean values and mean square errors (MSEs). The results obtained are presented in Table 6. It is evident from Table 6 that as the sample size increases, the mean value of the estimators approaches the original value of the respective parameters, and the MSEs of the estimators decrease accordingly.

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References

[1] Kumar, C.S. and Dharmaja, S.H.S., (2017). The exponentiated reduced Kies distribution: Properties and applications. Communications in Statistics-Theory and Methods, 46(17), pp.8778-8790.

[2] Kumar, C.S. and Dharmaja, S.H.S. (2013). On reduced Kies distribution. Collection of Recent Statistical Methods and Applications, (C.S. Kumar, M. Chacko and E.I.A. Sathar, Eds.), Department of Statistics, University of Kerala Publishers, Trivandrum,111– 123.